Canonical formalism for the Born-Infeld particle

1998 J. Phys. A: Math. Gen. 31 5775

(http://iopscience.iop.org/0305-4470/31/27/008)

J. Phys. A: Math. Gen. **31** (1998) 5775–5786. Printed in the UK PII: S0305-4470(98)91068-X

Canonical formalism for the Born–Infeld particle

Dariusz Chruściński†

Fakultät für Physik, Universität Freiburg, Hermann-Herder-Str. 3, D-79104 Freiburg, Germany

Received 27 January 1998

Abstract. It has been shown that the nonlinear Born–Infeld field equations supplemented by the 'dynamical condition' (certain boundary conditions for the field along a particle trajectory) define perfectly deterministic theory, i.e. the particle trajectory is determined without any equations of motion. In the present paper we show that this theory possesses mathematically consistent Lagrangian and Hamiltonian formulations. Moreover, it turns out that the 'dynamical condition' is already present in the definition of the physical phase space and, therefore, it is a basic element in the theory.

1. Introduction

Born–Infeld electrodynamics [2] were proposed in the 1930s as an alternative for Maxwell theory (see also [3] for a useful review). The main motivation of Born and Infeld was to construct a theory which had classical solutions representing electrically charged particles with finite self-energy. Born–Infeld theory indeed admits such solutions (recently Gibbons [4] proposed to call them 'BIons'). However, after Dirac's paper [5] on a classical electron and the birth of quantum electrodynamics in the 1940s Born–Infeld theory was almost totally forgotten for a long time.

Recently, there is a new interest in this theory due to investigations in string theory. It turns out that some very natural objects in this theory, so-called D-branes, are described by a kind of nonlinear Born–Infeld action (see, e.g., [4, 6]). Moreover, due to the remarkable interest in field and string theory dualities [7], the duality invariance of Born–Infeld electrodynamics has been studied in great detail [8] (actually this invariance was already observed by Schrödinger [9]).

Our motivation to study Born–Infeld electrodynamics is not due to string theory but to the dynamics of a classical point charge. There are the following reasons to consider nonlinear electromagnetism: it is well known that Maxwell electrodynamics when applied to point-like objects is inconsistent (see [10] for the review). This inconsistency originates in the infinite self-energy of the point charge. In Born–Infeld theory this self-energy is already finite. Therefore, one may hope that in the theory which gives this quantity a finite value it would be possible to describe particle self-interaction in a consistent way. Moreover, the assumption, that the theory is effectively nonlinear in the vicinity of the charged particle is very natural from the physical point of view which we have already learned from quantum electrodynamics (Born tried to make contact with quantum field theory by identifying the Born–Infeld Lagrangian as an effective Euler–Heisenberg Lagrangian [11]). It has been

† On leave from the Institute of Physics, Nicholas Copernicus University, ul. Grudziądzka 5/7, 87-100 Toruń, Poland.

shown [12] that the effective Lagrangian can coincide with those of Born and Infeld up to six-photon interaction terms only.

We consider a very specific model of nonlinear theory because, among other nonlinear theories of electromagnetism, Born–Infeld theory possesses very distinguished physical properties [13]. For example, it is the only causal spin-1 theory [14] (apart from the Maxwell one). Recently, Born–Infeld electrodynamics was successfully applied [15] as a model for the generation of multipole moments of charged particles.

In our previous paper [1] we studied the electrodynamics of a point charge in Born–Infeld nonlinear theory. Due to the nonlinearity of the field equations it is impossible to derive separate equations of motion for the charged particle corresponding, for example, to the celebrated Lorentz–Dirac equation in the Maxwell case. Could we, therefore, determine a particle trajectory without equations of motion? We showed [1] that it is in fact possible. Analysing the interaction between charged particles and nonlinear electromagnetism we proved that the conservation of total four-momentum of the composed (particle + field) system is equivalent to the certain boundary condition for the Born–Infeld field which has to be satisfied along the particle trajectory. We call it a 'dynamical condition' (formula (16) in the present paper) because, roughly speaking, it replaces the particle equations of motion. Field equations supplemented by this condition define perfectly deterministic theory, i.e. the initial data for the particle and field uniquely determine the evolution of the system.

In the present paper we show that the theory derived in [1] possesses consistent Lagrangian and Hamiltonian structures. It is important because any reasonable physical theory could be formulated this way. Therefore, we expect that Born–Infeld electrodynamics completed by the dynamical condition (16) is no exception to this rule. Moreover, we claim that the mathematically well defined canonical structure is a necessary condition for the theory to be consistent. It should be stressed that our model of a charged particle is different from that of Born and Infeld, i.e. our particle is not a purely electromagnetical 'BIon'. Nevertheless, we call it a 'Born–Infeld particle'. Particle mass, which appears, for example, in formula (15), could be interpreted as an effective mass, i.e. a 'mechanical' mass completed by 'radiative corrections' which are due to the electromagnetic interaction. The finite self-energy of a charge (i.e. the mass of its Coulomb field) is already contained in the field energy. Due to the nonlinearity of the theory there is no way to separate this self-energy from the total field energy (in Maxwell theory this separation enables one to perform the mass renormalization [10]). It turns out that the theory with a purely electromagnetical charged particle (contrary to the one considered in the present paper) does not have any consistent Hamiltonian formulation (this observation was made long ago by Pryce [16]).

The paper is organized as follows: in the next section we briefly sketch the main results of [1]. In section 3 a second-order particle Lagrangian is constructed and it is proved that the corresponding Euler–Lagrange equations are equivalent to our dynamical condition. Then in section 4 we present a Hamiltonian structure together with a Poisson bracket in section 5. Finally, we make some conclusions and outline possible generalizations.

2. Dynamical condition

In the present section we briefly sketch the main result presented in [1]. The Born–Infeld nonlinear electrodynamics [2] is based on the following Lagrangian (we use the Heaviside–Lorentz system of units with the velocity of light $c = 1$)

$$\mathcal{L}_{\rm BI} := \sqrt{-\det(b\eta_{\mu\nu})} - \sqrt{-\det(b\eta_{\mu\nu} + F_{\mu\nu})} = b^2(1 - \sqrt{1 - 2b^{-2}S - b^{-4}P^2}) \qquad (1)$$

where $\eta_{\mu\nu}$ denotes the Minkowski metric with the signature $(-, +, +, +)$. The standard Lorentz invariants S and P are defined by: $S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $P = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}$ ($\tilde{F}^{\mu\nu}$ denotes the dual tensor). The arbitrary parameter 'b' has the dimension of a field strength (Born and Infeld called it the *absolute field*) and it measures the nonlinearity of the theory. In the limit $b \to \infty$ the Lagrangian $\mathcal{L}_{\rm BI}$ tends to the Maxwell Lagrangian S.

Adding to (1) the standard electromagnetic interaction term '$j^\mu A_\mu$' we may derive the inhomogeneous field equations

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{2}$$

$$\partial_\mu G^{\mu\nu} = -j^\nu \tag{3}$$

where $G^{\mu\nu} := -2\partial\mathcal{L}_{\rm BI}/\partial F_{\mu\nu}$. The three-dimensional electric and magnetic fields are defined in a standard way

$$E^k = F^{0k} \qquad B^k = \tfrac{1}{2}\epsilon^{kij}F_{ij} \tag{4}$$

$$D^k = G^{0k} \qquad H^k = \tfrac{1}{2}\epsilon^{kij}F_{ij}. \tag{5}$$

Equations (2) and (3) rewritten in three-dimensional notation have the familiar form

$$\dot{\boldsymbol{B}} = -\nabla \times \boldsymbol{E} \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{6}$$

$$\dot{\boldsymbol{D}} = \nabla \times \boldsymbol{H} - \boldsymbol{j} \qquad \nabla \cdot \boldsymbol{D} = j^0 \tag{7}$$

with $j^\mu = (j^0, \boldsymbol{j})$. Let us observe that equations (6) and (7) have formally the same form as Maxwell equations. What makes the theory effectively nonlinear are the constitutive relations, i.e. relations between inductions $(\boldsymbol{D}, \boldsymbol{B})$ and intensities $(\boldsymbol{E}, \boldsymbol{H})$

$$\boldsymbol{E}(\boldsymbol{D}, \boldsymbol{B}) = \frac{1}{b^2 R}[(b^2 + \boldsymbol{B}^2)\boldsymbol{D} - (\boldsymbol{D}\boldsymbol{B})\boldsymbol{B}] \tag{8}$$

$$\boldsymbol{H}(\boldsymbol{D}, \boldsymbol{B}) = \frac{1}{b^2 R}[(b^2 + \boldsymbol{D}^2)\boldsymbol{B} - (\boldsymbol{D}\boldsymbol{B})\boldsymbol{D}] \tag{9}$$

with $R := \sqrt{1 + b^{-2}(\boldsymbol{D}^2 + \boldsymbol{B}^2) + b^{-4}(\boldsymbol{D} \times \boldsymbol{B})^2}$.

Now, let us assume that the external current j^μ in (3) is produced by a point-like particle moving along the time-like trajectory ζ. Due to the nonlinearity, the system (2) and (3) is very complicated to analyse. In particular, contrary to the Maxwell case, we do not know the general solution to the inhomogeneous Born–Infeld field equations. The main idea of [1] (it was proposed in the Maxwell case in [17]) was as follows: instead of solving distribution equations (2) and (3) on the entire Minkowski spacetime $\mathcal{M}$ let us treat them as a boundary problem in the region $\mathcal{M}_\zeta := \mathcal{M} - \{\zeta\}$, i.e. outside the trajectory. Obviously, in order to well-pose the problem we have to find an appropriate boundary condition which has to be satisfied along ζ, i.e. on the boundary $\partial\mathcal{M}_\zeta$.

To find this boundary condition we have analysed the asymptotic behaviour of the fields in the vicinity of a charged particle. The simplest way to do this is to use the particle rest frame. Let r denote the radial coordinate, i.e. a distance from a particle in its rest frame. Any vector field $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$ may be formally expanded in the vicinity of a charge

$$\boldsymbol{F}(\boldsymbol{r}) = \sum_n r^n \boldsymbol{F}_{(n)} \tag{10}$$

where the vectors $\boldsymbol{F}_{(n)}$ do not depend on r. The crucial observation is that the most singular part of the $\boldsymbol{D}$ field behaves as

$$\boldsymbol{D}_{(-2)} = \frac{e\mathcal{A}}{4\pi}\frac{\boldsymbol{r}}{r} \tag{11}$$

where, due to the Gauss law, the monopole part of the r-independent function $\mathcal{A}$ equals 1. Note that in the Maxwell case $\mathcal{A} \equiv 1$. Using (11) it was shown [1] that

$$\boldsymbol{H} \sim r^{-1} \qquad \boldsymbol{E} \sim r^{0} \qquad \boldsymbol{B} \sim r.$$

Moreover, the following theorem was proved.

Theorem 1. Any regular solution of Born–Infeld field equations with point-like external current satisfies

$$\boldsymbol{E}^{\mathrm{T}}_{(1)} = \frac{be}{4|e|}(3\boldsymbol{a} - r^{-2}(\boldsymbol{a}\boldsymbol{r})\boldsymbol{r}) \tag{12}$$

where $\boldsymbol{E}^{\mathrm{T}}$ stands for the transversal part of $\boldsymbol{E}$ and $\boldsymbol{a}$ denotes the particle acceleration.

Therefore, when the particle trajectory is *a priori* given, the hyperbolicity of (2) and (3) implies the following.

Theorem 2. The mixed (initial-boundary) value problem for the Born–Infeld equations in $\mathcal{M}_\zeta$ with (12) playing the role of boundary condition on $\partial\mathcal{M}_\zeta$ has a unique solution.

The above theorem is no longer true when we consider a particle as a dynamical object. Choosing the particle position $\boldsymbol{q}$ and velocity $\boldsymbol{v}$ as the Cauchy data for the particle dynamics let us observe that despite the fact that the time derivatives $(\dot{\boldsymbol{D}}, \dot{\boldsymbol{B}}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{v}})$ of the Cauchy data are uniquely determined by the data themselves, the evolution of the composed system is not uniquely determined. Indeed, $\dot{\boldsymbol{D}}$ and $\dot{\boldsymbol{B}}$ are given by the field equations, $\dot{\boldsymbol{q}} = \boldsymbol{v}$ and $\dot{\boldsymbol{v}}$ may be calculated from (12). Nevertheless, the initial value problem is not well-posed: keeping the same initial data, the particle trajectory can be modified almost at will. This is due to the fact, that (12) no longer plays the role of a boundary condition because we use it as a dynamical equation to determine $\boldsymbol{a}$. Therefore, a new boundary condition is necessary.

It was shown in [1] that this missing condition is implied by the conservation law of the total four-momentum for the 'particle + field' system

$$\dot{p}^{\mu}(t) = 0 \tag{13}$$

where p^{μ} stands for the four-momentum in any Lorentzian (laboratory) frame. Due to the field equations only three among four equations (13) are independent, i.e. the conservation of three-momentum

$$\dot{\boldsymbol{p}}(t) = 0 \tag{14}$$

implies the energy conservation. It was shown in [1] that (14) is equivalent to the following Newton-like equation

$$ma_k = \frac{|e|b}{3}\mathcal{A}_k \tag{15}$$

where $\mathcal{A}_k$ is the dipole part of $\mathcal{A}$, i.e. $\mathrm{DP}(\mathcal{A}) =: \mathcal{A}_k x^k/r$. This equation looks formally like a standard Newton equation. However, it could not be interpreted as the Newton equation because its right-hand side is not *a priori* given (it must be calculated from field equations).

To correctly interpret (15) we have to take into account (12). Now, by calculating $\boldsymbol{a}$ in terms of $\boldsymbol{E}^{\mathrm{T}}_{(1)}$ and inserting into (15) we obtain the following relation between the radial parts of $\boldsymbol{E}^{\mathrm{T}}_{(1)}$ and $\boldsymbol{D}_{(-2)}$

$$\mathrm{DP}(4r_0^4(\boldsymbol{E}^{\mathrm{T}}_{(1)})^r - \lambda_0(\boldsymbol{D}_{(-2)})^r) = 0 \tag{16}$$

where $r_0 = \sqrt{|e|/4\pi b}$ and $\lambda_0 = e^2/6\pi m$. All constants (like electric charge or particle mass) enter into the dynamical condition (16) via two characteristic scales r_0 and λ_0. Obviously r_0 measures the nonlinearity of the Born–Infeld theory (in the Maxwell case $r_0 \equiv 0$). The second scale λ_0 appears in any electrodynamics of charged particles, for example, it appears in the Lorentz–Dirac equation: $a^\mu = (e/m)F^{\mu\nu}_{\text{ext}}u_\nu + \lambda_0(\dot{a}^\mu - a^2u^\mu)$. The main result of [1] consists of the following theorem.

Theorem 3. Born–Infeld field equations supplemented by the dynamical condition (16) define perfectly deterministic theory, i.e. the initial data for field and particle uniquely determine the entire evolution of the system.

3. Lagrangian structure

In this section we show that the dynamical condition (16) may be derived from the mathematically well-defined variational principle. Guided by the example of Maxwell theory one could guess that such a principle should be based on the following Lagrangian

$$L_{\text{total}} = L_{\text{field}} + L_{\text{particle}} + L_{\text{int}} \tag{17}$$

with L_{field} given by (1), $L_{\text{particle}} = -m\gamma^{-1}$ ($\gamma^{-1} \equiv \sqrt{1-v^2}$) and $L_{\text{int}} = A_\mu j^\mu$. Now, by varying L_{total} with respect to A_μ one obviously reproduces (2) and (3). However, the variation with respect to a particle trajectory leads to the standard Lorentz equation

$$ma^\mu = eF^{\mu\nu}u_\nu. \tag{18}$$

However, despite the fact that $F^{\mu\nu}$ is bounded, it is not regular at the particle position and, therefore, the right-hand side of (18) is not well defined. This was already our motivation to find the mathematically well-defined dynamical condition (16) which replaces the ill-defined equations of motion (18).

Therefore, the variational principle based on (17) is not well defined. To find the correct principle we consider more carefully the field dynamics with respect to an arbitrary moving observer (see [18] and [19] for a general discussion). Consider an observer moving along an arbitrary (time-like) trajectory ζ. At each point $(t, \boldsymbol{q}(t)) \in \zeta$, let Σ_t denote a three-dimensional hyperplane orthogonal to the four-velocity vector $U = (u^\mu)$. Choose on Σ_t any system (x^i) of Cartesian coordinates, such that an observer is located at its origin. This system is not uniquely defined because on each Σ_t we still have the freedom of an $O(3)$-rotation. Let L denote the boost relating to the laboratory time axis $\partial/\partial y^0$ with the co-moving observer proper time axis U. Next, define the position of the $\partial/\partial x^k$ axis on Σ_t by transforming the corresponding $\partial/\partial y^k$ axis of the laboratory frame by the same boost. It is easy to verify that one obtains

$$y^0(t, \boldsymbol{x}) := t + \gamma(t)x^l v_l(t) \tag{19}$$

$$y^k(t, \boldsymbol{x}) := q^k(t) + \mathsf{L}^k_l(t)x^l \tag{20}$$

where the boost

$$\mathsf{L}^k_l := \delta^k_l + \gamma(1+\gamma^{-1})^{-1}v^k v_l. \tag{21}$$

The flat Minkowski metric tensor has in the new coordinates $(x^0 = t, x^k)$ the following form: $g_{kl} = \delta_{kl}$, whereas the lapse function N and the purely rotational shift vector $\boldsymbol{N}$ read

$$N = \frac{1}{\sqrt{-g^{00}}} = \gamma^{-1}(1 + \boldsymbol{ax}) \tag{22}$$

$$N_m = g_{0m} = \gamma^{-1}(\boldsymbol{\omega} \times \boldsymbol{x})_m \tag{23}$$

where $\boldsymbol{a}$ is the observer acceleration vector in the co-moving frame

$$a^i = \gamma^2 \mathsf{L}^i_l \dot{v}^l \tag{24}$$

and

$$\boldsymbol{\omega} = \gamma^2 (1 + \gamma^{-1})^{-1} \boldsymbol{v} \times \dot{\boldsymbol{v}}. \tag{25}$$

The field equations have in the co-moving frame the following form (cf. a general discussion in [20])

$$\dot{\boldsymbol{D}} = \nabla \times (N\boldsymbol{H}) + (\boldsymbol{N} \cdot \nabla)\boldsymbol{D} - (\boldsymbol{D} \cdot \nabla)\boldsymbol{N} \tag{26}$$

$$\dot{\boldsymbol{B}} = -\nabla \times (N\boldsymbol{E}) + (\boldsymbol{N} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{N}. \tag{27}$$

In [1] we used a simplified (so-called Fermi–Walker) frame in which the shift vector $\boldsymbol{N} = 0$. However, such a frame is non-local in time and, therefore, one cannot use it to define the Hamiltonian structure in a consistent way. We observe, that the field evolution given by (26) and (27) is a superposition of the following three transformations:

- time-translation in the direction of the observer four-velocity U;
- a boost in the direction of the observer acceleration $\boldsymbol{a}$;
- purely spatial $O(3)$-rotation around the vector $\boldsymbol{\omega}$.

It is, therefore, obvious that this evolution is generated by the following generator

$$H_{\text{field}} := -\gamma^{-1}(\mathcal{P}^0 + \boldsymbol{a}\boldsymbol{\mathcal{R}} - \boldsymbol{\omega}\boldsymbol{\mathcal{S}}) \tag{28}$$

where $\mathcal{P}^0$ is the field energy, $\boldsymbol{\mathcal{R}}$ is the static moment and $\boldsymbol{\mathcal{S}}$ denotes the angular momentum. These quantities, together with the field momentum $\boldsymbol{\mathcal{P}}$, are defined in the observer rest frame via a symmetric energy–momentum tensor

$$T^{\mu}_{\nu} = \delta^{\mu}_{\nu} \mathcal{L}_{\text{BI}} - \frac{\partial \mathcal{L}_{\text{BI}}}{\partial S} F^{\mu}_{\lambda} F^{\lambda}_{\nu} - \frac{\partial \mathcal{L}_{\text{BI}}}{\partial P} F^{\mu}_{\lambda} \tilde{F}^{\lambda}_{\nu} \tag{29}$$

in the standard way:

$$\mathcal{P}^0(t) = \int_{\Sigma_t} T^{00}\, \mathrm{d}^3x \tag{30}$$

$$\mathcal{P}_k(t) = \int_{\Sigma_t} T^0_k\, \mathrm{d}^3x \tag{31}$$

$$\mathcal{R}_k(t) = \int_{\Sigma_t} x_k T^{00}\, \mathrm{d}^3x \tag{32}$$

$$\mathcal{S}_k(t) = \int_{\Sigma_t} \epsilon_{klm} x^l T^{0m}\, \mathrm{d}^3x \tag{33}$$

where Σ_t denotes a rest hyperplane intersecting the observer trajectory at time t. The relativistic factor γ^{-1} in (28) corresponds to the fact that we do not use the proper time along the observer trajectory ζ but the laboratory time. The '$-$' is chosen for future convenience. We stress that H_{field} plays the role of a Hamiltonian (with negative sign) for any relativistic Lagrangian field theory. In the case of electrodynamics (Maxwell or general nonlinear theory) the corresponding Hamilton equations are given by (26) and (27).

Now, let us add to this picture a charged particle by replacing the field energy $\mathcal{P}^0$ by a total 'particle + field' energy $\mathcal{H} = m + \mathcal{P}^0$. Because the energy–momentum tensor (29) varies in the vicinity of a particle as r^{-2} (in Maxwell theory as r^{-4}), the Poincaré generators (30)–(33) are well defined. Contrary to the Maxwell case no renormalization is necessary. Obviously, the particle static moment and angular momentum vanish in its rest frame. We define the new generator

$$L_H := -\gamma^{-1}(\mathcal{H} + \boldsymbol{a}\boldsymbol{\mathcal{R}} - \boldsymbol{\omega}\boldsymbol{\mathcal{S}}). \tag{34}$$

It turns out that the above generator contains all the information about particle dynamics. This is due to the following theorem.

Theorem 4. The generator L_H defines the second-order particle Lagrangian, i.e. varying it with respect to the particle trajectory one recovers the dynamical condition (16).

Proof. The Euler–Lagrange equations for a second-order Lagrangian read

$$\dot{\boldsymbol{p}} = \frac{\partial L_H}{\partial \boldsymbol{q}} \tag{35}$$

where the momentum $\boldsymbol{p}$ canonically conjugated to the particle position $\boldsymbol{q}$ is defined by

$$\boldsymbol{p} := \frac{\partial L_H}{\partial \boldsymbol{v}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_H}{\partial \dot{\boldsymbol{v}}}\right) \tag{36}$$

(see [21] for a general discussion of higher-order Lagrangians; a review of a second-order case may be found in [18] and [19]). To calculate $\boldsymbol{p}$ one needs time derivatives of $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{S}}$. Using the field equations (26) and (27) and the asymptotic conditions presented in the previous section one gets

$$\dot{\boldsymbol{\mathcal{R}}} = \gamma^{-1}(\boldsymbol{\mathcal{P}} - \boldsymbol{a} \times \boldsymbol{\mathcal{S}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{R}}) \tag{37}$$

$$\dot{\boldsymbol{\mathcal{S}}} = \gamma^{-1}(\boldsymbol{a} \times \boldsymbol{\mathcal{R}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{S}}). \tag{38}$$

Therefore, one obtains the following formula

$$p_k = \gamma v_k \mathcal{H} + \mathbf{L}_k^l \mathcal{P}_l + \mathbf{A}_k^l \mathcal{R}_l + \mathbf{B}_k^l \mathcal{S}_l \tag{39}$$

with

$$\mathbf{A}_k^l = \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma^{-1}\frac{\partial a^l}{\partial \dot{v}^k}\right) - \frac{\partial(\gamma^{-1}a^l)}{\partial v^k} - \gamma^{-2}\epsilon^{il}_{m}\left(\frac{\partial a^m}{\partial \dot{v}^k}\omega_i + \frac{\partial \omega^m}{\partial \dot{v}^k}a_i\right)$$

$$\mathbf{B}_k^l = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma^{-1}\frac{\partial \omega^l}{\partial \dot{v}^k}\right) + \frac{\partial(\gamma^{-1}\omega^l)}{\partial v^k} + \gamma^{-2}\epsilon^{il}_{m}\left(\frac{\partial a^m}{\partial \dot{v}^k}a_i + \frac{\partial \omega^m}{\partial \dot{v}^k}\omega_i\right).$$

After straightforward (but tedious) algebra one finds: $\mathbf{A}_k^l = \mathbf{B}_k^l = 0$ (actually, there is a simpler way to observe that both $\mathbf{A}_k^l$ and $\mathbf{B}_k^l$ vanish. Due to the relativistic invariance one could calculate these quantities for $\boldsymbol{v} = 0$ and then transform the results by an appropriate Lorentz boost. However, for $\boldsymbol{v} = 0$ one immediately sees that $\mathbf{A}_k^l = \mathbf{B}_k^l = 0$ and, obviously, it is also true for an arbitrary $\boldsymbol{v}$.). Finally, one obtains

$$p_k = \gamma v_k \mathcal{H} + \mathbf{L}_k^l \mathcal{P}_l. \tag{40}$$

However, (40) is a total 'particle + field' momentum in the laboratory frame. Therefore, the Euler–Lagrange equations (35) reduce to the conservation law of the total momentum (since the right-hand side of (35) vanishes in our case) and it was proved in [1] that it is equivalent to the dynamical condition (16). This ends the proof. □

4. Hamiltonian

To find the corresponding Hamiltonian structure of this theory one has to perform the Legendre transformation to L_H. Let $\boldsymbol{\pi}$ denote the momentum canonically conjugated to the particle velocity $\boldsymbol{v}$, i.e.

$$\boldsymbol{\pi} := \frac{\partial L_H}{\partial \dot{\boldsymbol{v}}}. \tag{41}$$

We observe that due to the fact that L_H is linear in $\dot{\boldsymbol{v}}$ (see (24) and (25)) it is impossible to invert (41) (i.e. to calculate $\dot{\boldsymbol{v}}$ in terms of $\boldsymbol{\pi}$) and, therefore, the Legendre transformation is singular. It means that in the Hamiltonian framework the phase space of our system

$$\mathcal{P} = (\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{v}, \boldsymbol{\pi}; \boldsymbol{D}, \boldsymbol{B})$$

is subject to some constraints. To find these constraints let us apply the standard Dirac–Bergmann procedure [22] (see also [23]). The phase space $\mathcal{P}$ is endowed with the following canonical Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\} = \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}} \cdot \frac{\partial \mathcal{G}}{\partial \boldsymbol{p}} + \frac{\partial \mathcal{F}}{\partial \boldsymbol{v}} \cdot \frac{\partial \mathcal{G}}{\partial \boldsymbol{\pi}} + \int_\Sigma \frac{\delta \mathcal{F}}{\delta \boldsymbol{D}} \cdot \nabla \times \frac{\delta \mathcal{G}}{\delta \boldsymbol{B}} \, \mathrm{d}^3x - (\mathcal{F} \rightleftharpoons \mathcal{G}). \tag{42}$$

Now, the (unreduced) Hamiltonian on $\mathcal{P}$ is defined by

$$H(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{v}, \boldsymbol{\pi}; \boldsymbol{D}, \boldsymbol{B}) = \boldsymbol{p}\boldsymbol{v} + \boldsymbol{\pi}\dot{\boldsymbol{v}} - L_H. \tag{43}$$

Therefore, the primary and secondary constraints read

$$\phi_k^{(1)} := \pi_k - \frac{\partial L_H}{\partial \dot{v}^k} \approx 0 \tag{44}$$

$$\phi_k^{(2)} := \{\phi_k^{(1)}, H\} = -p_k + \gamma v_k \mathcal{H} + \mathsf{L}_k^l \mathcal{P}_l \approx 0 \tag{45}$$

where the symbol '≈' refers to the 'weak equality'. We see that the secondary constraints $\phi_k^{(2)} = 0$ reproduce (40). Using (44) and (45) we get

$$H = \boldsymbol{p}\boldsymbol{v} + \gamma^{-1}\mathcal{H} + \dot{v}^k \phi_k^{(1)} \tag{46}$$

and $\dot{v}^k$ now plays the role of a Lagrange multiplier. Let us continue with the Dirac–Bergmann procedure and look for the tetriary constraints

$$\phi_k^{(3)} := \{\phi_k^{(2)}, H\} \approx 0 \tag{47}$$

with H given by (46). One may show (following the calculations in [1]) that (47) implies

$$\dot{v}^k = \gamma^{-2} \frac{|e|b}{3m} (\mathsf{L}^{-1})_l^k \mathcal{A}^l \tag{48}$$

which is equivalent to (15). Together with (12) it implies our dynamical condition (16). Therefore, the consistency of the theory requires that the dynamical condition has to be already present in the definition of the physical phase space. This way the tetriary constraints are identically satisfied and the constraint algorithm stops at this point. Thus, the Lagrange multiplier $\dot{\boldsymbol{v}}$ in (46) is, therefore, already known which means that the constraints $\phi_k^{(1)}$ and $\phi_k^{(2)}$ are of the second class (see the next section).

Now, on the reduced phase space, i.e. $\mathcal{P}$ subjected to (44) and (45) (and to the dynamical condition (16)) the reduced Hamiltonian

$$H^*(\boldsymbol{q}, \boldsymbol{v}; \boldsymbol{D}, \boldsymbol{B}) = \gamma(\mathcal{H} + \boldsymbol{v}\boldsymbol{\mathcal{P}}) \tag{49}$$

is equal to the total energy of the composed system in the laboratory frame. In deriving (49) we chose as independent variables (in the particle sector) the particle position $\boldsymbol{q}$ and velocity $\boldsymbol{v}$ (this choice is strongly suggested by the form of constraints equations). Obviously, this parametrization is not unique. For example, we could take as independent variables $\boldsymbol{q}$ and $\boldsymbol{p}$ as well. One could show (see [19]) that

$$\boldsymbol{v} = \frac{[\boldsymbol{p}(\boldsymbol{p} - \boldsymbol{\mathcal{P}})](\sqrt{\mathcal{H}^2 + \boldsymbol{p}^2 - \boldsymbol{\mathcal{P}}^2} - \mathcal{H})}{[\boldsymbol{p}(\boldsymbol{p} - \boldsymbol{\mathcal{P}})]^2 + \mathcal{H}^2(\boldsymbol{p} - \boldsymbol{\mathcal{P}})^2} (\boldsymbol{p} - \boldsymbol{\mathcal{P}}) \tag{50}$$

and, therefore

$$H^*(\boldsymbol{q}, \boldsymbol{p}; \boldsymbol{D}, \boldsymbol{B}) = \sqrt{\mathcal{H}^2 + \boldsymbol{p}^2 - \boldsymbol{\mathcal{P}}^2}. \tag{51}$$

Obviously, numerically $H^*(\boldsymbol{q}, \boldsymbol{v}; \boldsymbol{D}, \boldsymbol{B}) = H^*(\boldsymbol{q}, \boldsymbol{p}; \boldsymbol{D}, \boldsymbol{B})$. We see that for a free particle, i.e. $e = 0$, $\mathcal{H} = m$ and $\boldsymbol{\mathcal{P}} = 0$, the complicated formula (50) reduces to the relativistic relation between the particle velocity and momentum: $\boldsymbol{v} = \boldsymbol{p}/\sqrt{m^2 + \boldsymbol{p}^2}$, and formula (51) reproduces the relativistic particle energy: $\mathcal{E}(\boldsymbol{p}) = \sqrt{m^2 + \boldsymbol{p}^2}$.

5. Poisson bracket

In this section we reduce the Poisson bracket (42) (defined on $\mathcal{P}$) on the reduced phase space

$$\mathcal{P}^* = (\boldsymbol{q}, \boldsymbol{v}; \boldsymbol{D}, \boldsymbol{B}).$$

This is possible because, as one can easily check, constraints (44) and (45) are of the second class

$$\begin{aligned} &\{\phi_k^{(1)}, \phi_l^{(1)}\} = 0 \\ &\{\phi_k^{(1)}, \phi_l^{(2)}\} = -\gamma m(g_{kl} + \gamma^2 v_k v_l) \\ &\{\phi_k^{(2)}, \phi_l^{(2)}\} = \frac{\gamma |e| b}{3}(v_k \mathcal{A}_l - v_l \mathcal{A}_k). \end{aligned} \tag{52}$$

Therefore, due to the rules of the Dirac–Bergmann procedure [22] we obtain the following formula for the reduced Poisson (or Dirac) bracket on $\mathcal{P}^*$

$$\{\mathcal{F}, \mathcal{G}\}^* = \{\mathcal{F}, \mathcal{G}\} + \mathbf{X}^{kl}(\{\mathcal{F}, \phi_k^{(2)}\}\{\phi_l^{(1)}, \mathcal{G}\} - \{\mathcal{F}, \phi_k^{(1)}\}\{\phi_l^{(2)}, \mathcal{G}\}) - \mathbf{Y}^{kl}\{\mathcal{F}, \phi_k^{(1)}\}\{\phi_l^{(1)}, \mathcal{G}\} \tag{53}$$

where

$$\mathbf{X}^{kl} = \frac{g^{kl} - v^k v^l}{\gamma m} \tag{54}$$

$$\mathbf{Y}^{kl} = \frac{|e| b}{3m^2\gamma^3}(v^k \mathcal{A}^l - v^l \mathcal{A}^k). \tag{55}$$

In particular, one easily shows that in the 'particle sector':

$$\begin{aligned} &\{q^k, q^l\}^* = 0 \\ &\{q^k, v^l\}^* = \mathbf{X}^{kl} \\ &\{v^k, v^l\}^* = \mathbf{Y}^{kl}. \end{aligned} \tag{56}$$

Others 'commutation relations' between variables parametrizing $\mathcal{P}^*$ may be easily obtained from (53).

Using (53) we are able to reproduce the dynamics of our theory

$$\dot{q}^k = \{q^k, H^*\}^* = v^k \tag{57}$$

$$\dot{v}^k = \{v^k, H^*\}^* = \gamma^{-2}\frac{|e| b}{3m}(\mathbf{L}^{-1})^k_l \mathcal{A}^l \tag{58}$$

and in the 'field sector' one easily finds that

$$\dot{D}^k = \{D^k, H^*\}^* \tag{59}$$

$$\dot{B}^k = \{B^k, H^*\}^* \tag{60}$$

are equivalent to (26) and (27) where $\dot{\boldsymbol{v}}$ is given now by (58). We observe, that (58) is nothing more than an identity because the dynamical condition (which is equivalent to (58) together with (12)) is already present in the definition of $\mathcal{P}^*$. This way we have proved the following.

Theorem 5. The triple $(\mathcal{P}^*, H^*, \{\,,\,\}^*)$ defines mathematically consistent canonical structure of a point-like charge particle interacting with nonlinear Born–Infeld electromagnetism.

As a simple implication one can prove the following theorem.

Theorem 6. Laboratory-frame Lorentz generators:

$$p^0 = \gamma(\mathcal{H} + v^l\mathcal{P}_l)$$
$$p_k = \gamma v_k\mathcal{H} + \mathsf{L}_k^l\mathcal{P}_l$$
$$r_k = \gamma^3(\mathsf{L}^{-1})_k^l\mathcal{R}_l + \gamma\epsilon_{klm}v^l\mathcal{S}^m + q_k p^0$$
$$s_k = \gamma^3(\mathsf{L}^{-1})_k^l\mathcal{S}_l - \gamma\epsilon_{klm}v^l\mathcal{R}^m + \epsilon_{klm}q^l p^m$$

form with respect to $\{\,,\,\}^*$ the Poincaré algebra.

This shows that the canonical structure of our theory is perfectly consistent with the relativistic invariance.

6. Concluding remarks

Finally, let us make the following remarks:

(1) a formalism applied in the present paper is perfectly gauge-invariant. There is no need to use a gauge potential to couple a particle to the field.

(2) The second-order particle Lagrangian L_H cannot be written in the form as in formula (17). In particular there is no 'interaction term' in L_H. All information about the interaction between a particle and fields is contained in the asymptotic conditions for the electromagnetic field in the vicinity of particle trajectory. We observe that L_H serves as a Lagrangian for a particle dynamics and a Hamiltonian for the field dynamics. Therefore, it is a non-trivial example of a so-called Routhian function known from analytical mechanics.

(3) A total four-momentum p^μ of the composed 'particle+field' system lies always (as it should) in the forward light-cone. In Maxwell theory the corresponding energy–momentum tensor is not integrable in the vicinity of a charged particle and, therefore, one has to perform an appropriate renormalization. The simple renormalization scheme proposed in [17] defines p^μ which does not satisfy this obvious property.

(4) The remarkable feature of the reduced Poisson bracket is that it is analytical at the point $e = 0$. In our opinion this property is important for the construction of the consistent electrodynamics of point-like objects. In the Maxwell case it is well known that all non-physical solutions to the Lorentz–Dirac equation are non-analytical at $e = 0$ [10]. It turns out in [19] that this non-analyticity is also present in the corresponding Hamiltonian framework where we do not have any equations of motion (there is an analogue of the dynamical condition). Namely, the Poisson bracket is non-analytical at $e = 0$. Therefore, the analyticity of the canonical structure seems to be an important ingredient to prove consistency of the theory. This point will be carefully discussed in the next paper.

(5) It is clear from (52) that in the case of a purely electromagnetical particle (i.e. when the effective mass $m = 0$) the constraints $\phi_k^{(1)}$ and $\phi_k^{(2)}$ are no longer of a second class—$\phi_k^{(1)}$ are first class. Therefore, they give rise to a certain gauge freedom and the reduced phase space $\mathcal{P}^* = (\boldsymbol{q}, \boldsymbol{v}; \boldsymbol{D}, \boldsymbol{B})$ is not a proper space of states in this case, i.e. the dynamics of our system cannot be consistently reduced on $\mathcal{P}^*$. This means that the data $(\boldsymbol{q}, \boldsymbol{v}; \boldsymbol{D}, \boldsymbol{B})$ does not determine the evolution uniquely (there is a gauge freedom). The observation that

the purely electromagnetical particles do not have consistent Hamiltonian formulation was made long ago by Pryce (see section 8 of [16]).

(6) Finally, we observe that variables $(\boldsymbol{q}, \boldsymbol{v}; \boldsymbol{D}, \boldsymbol{B})$ are highly non-canonical with respect to the reduced Poisson bracket. It would be interesting to find a canonical set.

The results obtained in the present paper could be generalized to include many charged particles, and to include Born–Infeld dyons, i.e. particles possessing both electric and magnetic charges. Work on these points is in progress.

Acknowledgments

The author would like to thank Professor J Kijowski for many discussions on the canonical formalism for classical field theories, Professor I Białynicki-Birula for pointing out [16] and Professor H Römer for his interest in this work. Financial support from the Alexander von Humboldt-Stiftung is gratefully acknowledged.

References

[1] Chruściński D 1998 *Phys. Lett.* **240A** 8
[2] Born M 1934 *Proc. R. Soc.* A **143** 410
Born M and Infeld L 1934 *Proc. R. Soc.* A **144** 425
[3] Białynicki–Birula I and Białynicka–Birula 1975 *Quantum Electrodynamics* (Oxford: Pergamon)
Białynicki–Birula I 1983 Nonlinear electrodynamics: variations on a theme of Born and Infeld *Quantum Theory of Particles and Fields* ed B Jancewicz and J Lukierski (Singapore: World Scientific) pp 31–48
[4] Gibbons G W 1997 Born–Infeld particles and Dirichlet p-branes *Preprint* hep-th/9709027
[5] Dirac P A M 1938 *Proc. R. Soc.* A **167** 148
[6] Bachas C 1997 (Half) a lecture on D-branes *Preprint* hep-th/9701019
Fradkin E S and Tseytlin A A 1985 *Phys. Lett.* **163B** 123
Perry M and Schwarz J H 1996 Interacting chiral gauge fields in six dimensions and Born–Infeld theory *Preprint* hep-th/9611065
[7] Olive D 1997 *Nucl. Phys. (Proc. Suppl.)* B **58** 43
[8] Gibbons G W and Rasheed D A 1995 *Nucl. Phys.* B **454** 185
Gibbons G W and Rasheed D A 1996 *Phys. Lett.* **365B** 46
Gaillard M K and Zumino B 1997 Self-duality in nonlinear electromagnetism *Preprint* hep-th/9705226
Bengtsson I 1996 Manifest duality in Born–Infeld theory *Preprint* hep-th/9612174
[9] Schrödinger E 1935 *Proc. R. Soc.* A **150** 466
[10] Rohrlich F 1965 *Classical Charged Particles* (Reading: Addison-Wesley)
[11] Heisenberg W and Euler H 1936 *Z. Phys.* **98** 714
Euler H and Kockel B 1935 *Naturwissenschaften* **23** 246
[12] Hagiwara T 1981 *Nucl. Phys.* B **189** 135
[13] Hagiwara T 1981 *J. Phys. A: Math. Gen.* **14** 3059
[14] Plebański J 1968 *Lectures on Non-linear Electrodynamics* (Copenhagen: Nordita)
[15] Chruściński D and Kijowski J 1997 *C. R. Acad. Sci., Paris* **324** Serié IIb 435
Chruściński D and Kijowski J 1998 *J. Phys. A: Math. Gen.* **31** 269
[16] Pryce H L M 1936 *Proc. R. Soc.* A **155** 597
[17] Kijowski J 1994 *Gen. Rel. Grav.* **26** 167
Kijowski J 1994 *Acta Phys. Polon.* A **85** 771
[18] Kijowski J and Chruściński D 1995 *Gen. Rel. Grav.* **27** 267
[19] Chruściński D 1998 *Rep. Math. Phys.* **41** 13
[20] Misner C, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman)
[21] Gitman D M and Tyutin I V 1990 *Quantization of Fields with Constraints* (Berlin: Springer)
[22] Dirac P A M 1950 *Can. J. Math.* **2** 129
Dirac P A M 1964 Lectures on quantum mechanics *Monograph Series* Belfer Graduate School of Science
Bergmann P 1961 *Rev. Mod. Phys.* **33** 510

[23] Hanson A J, Regge T and Teitelboim C 1976 *Constrained Hamiltonian Systems* (Rome: Academia Nazionale de Lincei)
Sundermeyer K 1982 *Constrained Dynamics (Lecture Notes in Physics 169)* (Berlin: Springer)
Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)